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THE HYDROMAGNETIC STABILITY OF A
TOROIDAL GAS DISCHARGE

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ABSTRACT

The stability of a toroidal gas discharge is examined by using an energy principle. In the equilibrium configuration the plasma is assumed to be perfectly conducting and to have constant pressure and density. It is found that the pinch can be stabilized under certain very restricted conditions. However, one of the conditions is that the aspect ratio be small so that a long thin torus will always be unstable in the limit.

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THE HYDROMAGNETIC STABILITY OF A TOROIDAL GAS DISCHARGE

1. Introduction.

In recent years many experiments on the pinch effect have been carried out using both cylindrical and toroidal geometries. Until now cylindrical gas discharges have been treated theoretically and it has been found [1] that the pinch can be stabilized under certain conditions. It is the purpose of this investigation to study the stability of a toroidal gas discharge and to get some indications of the influence of the curvature on the stability. This influence should be not negligible, if, as was the case in some experiments, the so-called aspect-ratio — the ratio of the small diameter of the torus to the large diameter — is not small compared to one.

Two different methods may be applied for the investigation of the stability of a hydromagnetic equilibrium configuration. The first method makes use of the normal mode analysis and with this method the stability of cylindrical gas discharges have been treated [1]. But for more complicated equilibrium

configurations it turns out that the application of an energy principle is more useful. This method is used in the following treatment. The main reason for this is that the mode analysis leads to a differential equation where the boundary conditions are quite complicated in the case of toroidal geometry. This complication can be avoided using the energy principle.

2. The Equilibrium Configuration.

We assume that the plasma, regarded as an ideal, compressible, perfectly conducting fluid, has the shape of a torus and is surrounded by a conductor, as indicated in Figure 1. In the equilibrium configuration, the pressures and density inside the plasma are taken to be constant; it is assumed that there are no electric currents flowing except on the surface of the plasma; and the magnetic field is taken to be of the form

$$\vec{B} = (0, b/r, 0) \quad \text{inside plasma} \tag{2.1a}$$

$$\vec{\hat{B}} = (\hat{B}_r, \hat{b}/r, \hat{B}_z) \quad \text{outside plasma} \tag{2.1b}$$

in which the components have been listed in the order, r-component, $\emptyset$-component, z-component, and in which b and $\hat{b}$ are constants. The coordinate system $(r, \emptyset, z)$ is cylindrical, as indicated in the figure.

In order that this be an equilibrium configuration,

the meridional external field, whose components are $\hat{B}_r$ and $\hat{B}_z$, must (jointly with the given azimuthal fields) provide the required confining pressures on the plasma. The field must be purely tangential. The pressure balance gives the relation

$$\frac{1}{2}(\hat{B}^2 - B^2) = p \qquad (2.2)$$

where p is the constant equilibrium pressure of the plasma. This external field, if extended in the surrounding space, would have various singularities, for example, on the axis. It must therefore be supposed that somewhere outside the apparatus, but not shown in the drawing, there are suitable current coils to produce this field. The conducting wall shown is supposed to have no effect on the equilibrium field. The cross-sections of the plasma surface and of the conducting wall are assumed to be circles lying on coordinate surfaces $\eta = \eta_o$ and $\eta = \eta_1$ in a toroidal coordinate system (η, ϕ, θ) and therefore not quite concentric. With this assumption, the conductor is not placed on a line of force of the vacuum magnetic field as the equilibrium condition requires. But we may assume that the equilibrium field has enough time to penetrate the conductor and that the conductor acts only on the perturbed field. This inconsistency and the perhaps improbable nature of

the current coils that would be required to produce the equilibrium field specified [4], are excused on the ground that the finer details of the equilibrium field probably have little effect on the hydromagnetic stability and would not be accurately known in a given experimental set-up anyway.

The toroidal coordinates are related to the cylindrical coordinates by the transformation

$$r = \frac{a \sinh \eta}{\cosh \eta - \cos \theta} \tag{2.3a}$$

$$\phi = \phi \tag{2.3b}$$

$$z = \frac{a \sin \theta}{\cosh \eta - \cos \theta} \tag{2.3c}$$

where $(\pm a, 0)$ are the poles of the coordinate system in the x,y-plane. From this we get the radius of the cross-section of the plasma

$$R_o = \frac{a}{\sinh \eta_o} \tag{2.4}$$

with its center

$$r_o = a \frac{\cosh \eta_o}{\sinh \eta_o} \tag{2.5}$$

and the radius of the cross-section of the conductor

$$R_1 = \frac{a}{\sinh \eta_1} . \tag{2.6}$$

We define the aspect ratio

$$\Delta = \frac{r_1}{R_o} = \cosh \eta_1 \, . \tag{2.7}$$

The equilibrium condition (2.2) is given in this coordinate system by

$$\frac{1}{2}\left\{(\hat{B}_\theta^2 + \frac{(\cosh \eta_o - \cos \theta)^2}{a^2 \sinh^2 \eta_o} (\hat{b}^2 - b^2)\right\} = p \, . \tag{2.8}$$

3. The Energy Principle.

As mentioned above we want to use an energy principle [2] for investigating the given equilibrium configuration. This energy principle says that the necessary and sufficient condition for stability is that the change in energy δW for a given displacement $\vec{\xi}$ must be positive for all $\vec{\xi}$.

The change in energy has been derived by Bernstein et. al. [2] and is given by

$$\delta W = \delta W^{(i)} + \delta W^{(e)} + \delta W^{(s)} \, . \tag{3.1}$$

Here $\delta W^{(i)}$ is the contribution from the plasma (the internal part)

$$\delta W^{(i)} = \frac{1}{2}\int [\vec{Q} \cdot \vec{Q} + \gamma p \, (\mathrm{div}\, \vec{\xi})^2] \, d\tau \tag{3.2}$$

assuming that the pressure is a constant and that no electrical currents are flowing in the plasma. The

volume integral has to be extended over the initial volume of the plasma. $\vec{Q}$ is given by

$$\vec{Q} = \text{curl } (\vec{\xi} \times \vec{B}) \tag{3.3}$$

and γ is the ratio of the specific heats. The contribution of the vacuum $\delta W^{(e)}$ (the external part) is given by

$$\delta W^{(e)} = \frac{1}{2} \int (\text{grad } \Phi)^2 \, d\tau \tag{3.4}$$

where Φ is the scalar potential of the perturbed vacuum field and the integration must be extended over the vacuum which is surrounded by the plasma and the conductor. The potential Φ must fulfill the two boundary conditions:

$$\frac{\partial \Phi}{\partial n} = \text{curl } (\vec{\xi} \times \vec{B}) \Big|_n \tag{3.5}$$

on the plasma-vacuum interface, where the subscript refers to the normal component and

$$\frac{\partial \Phi}{\partial n} = 0 \tag{3.6}$$

on the containing wall. $\partial \Phi / \partial n$ is the normal derivative of Φ. These two boundary conditions do not determine the potential uniquely in some cases, since Φ need not be a single-valued function necessarily [3]. This point will be discussed later. Finally, the contribution from the plasma-vacuum interface is given by

$$\delta W^{(s)} = \frac{1}{2} \int (\vec{n} \cdot \vec{\xi})^2 \left(\vec{n} \cdot \left\langle \operatorname{grad}(p + \tfrac{1}{2}|B|^2)\right\rangle\right) d\sigma \qquad (3.7)$$

where $\langle \ldots \rangle$ denotes the increment of a quantity across the boundary in the normal direction $\vec{n}$. The integration has to be extended over the plasma-vacuum interface.

Using the energy principle the equation of stability is reduced to an examination of the sign of δW for arbitrary displacements $\vec{\xi}$. To do this one must minimize δW. For this procedure we follow a method suggested by Bernstein et. al. [2] First $\delta W^{(i)}$ and $\delta W^{(e)}$ are minimized separately by prescribing $\vec{n} \cdot \vec{\xi}$ on the plasma-vacuum boundary.

Minimizing $\delta W^{(i)}$ leads to the Euler-Lagrange equation for $\vec{\xi}$

$$\gamma\, p \operatorname{grad} (\operatorname{div} \vec{\xi}) - \vec{B} \times \operatorname{curl} \vec{Q} = 0 . \qquad (3.8)$$

When $\vec{\xi}$ is chosen to satisfy (3.8), the integral $\delta W^{(i)}$ in (3.2) can be reduced by partial integrations to a surface integral.

$$[\delta W^{(i)}]_{min} = \frac{1}{2} \int (\vec{n} \cdot \vec{\xi}) \, [\gamma p \operatorname{div} \vec{\xi} - \vec{B} \cdot \vec{Q}] \, d\sigma \qquad (3.9)$$

The external part $\delta W^{(e)}$ is minimized by choosing Φ to satisfy the Laplace equation:

$$\Delta \Phi = 0 . \qquad (3.10)$$

When Φ is so chosen, we get

$$[\delta W^{(e)}]_{min} = \frac{1}{2} \int [\Phi \vec{n} \circ \text{grad } \Phi] \, d\sigma . \qquad (3.11)$$

The unit normal $\vec{n}$ points into the plasma here, and out of the plasma in (3.9).

In this way δW is now reduced to a surface integral over the plasma-vacuum interface. The process of minimization is completed by minimizing δW given by (3.1) and (3.7), (3.9) and (3.11) with respect to $(\vec{n} \circ \vec{\xi})$.

4. The Stability of a Cylinder.

Before we calculate δW for a torus we will show how δW can be obtained for a cylinder. In a later section we will compare the results of the cylinder with those of the torus.

In a cylindrical coordinate system (r,ϕ,z) the equilibrium field is given by

$$\vec{B} = (0,0,b_c) \qquad \text{inside plasma} \qquad (4.1a)$$

$$\hat{\vec{B}} = (0,\hat{B}_\phi R_0/r,\hat{b}_c) \qquad \text{outside plasma} \qquad (4.1b)$$

and the equilibrium condition is given by the pressure balance

$$\frac{1}{2} (\hat{B}_\phi^2 + \hat{b}_c^2 - b_c^2) = p . \qquad (4.2)$$

(In bending the cylinder into a torus, the z-component of the magnetic field becomes the ϕ-component and the

ϕ-component becomes the meridional component. See eq. (2.1) and (4.1).) $\vec{\xi}$ can be Fourier analyzed in the form

$$\vec{\xi} = \vec{\xi}(r)\, e^{i(a\phi+kz)} \quad . \qquad (4.3)$$

Now the problem splits into two branches according as $k \neq 0$ or $k = 0$.

(a) $k \neq 0$:

From (3.8) it follows from scalar multiplication by $\vec{B}$ that

$$\operatorname{div} \vec{\xi} = 0 \quad . \qquad (4.4)$$

Introducing

$$\Psi = -(\vec{B} \circ \vec{Q}) / b_c^2 \qquad (4.5)$$

(3.8) gives

$$\Delta \Psi = 0 \qquad (4.6)$$

and

$$(\vec{n} \circ \vec{\xi}) = \frac{1}{k^2} (\vec{n} \circ \operatorname{grad} \Psi) \quad . \qquad (4.7)$$

Using (4.5) and (4.7) we get for (3.9):

$$[\delta W^{(i)}]_{min} = \frac{1}{2} \frac{b_c^2}{k^2} \int \Psi (\vec{n} \cdot \operatorname{grad} \Psi)\, d\sigma \, . \qquad (4.8)$$

For calculating $\delta W^{(s)}$ (eq. (3.7)), the differences of the total pressure gradient across the surface is needed. Using (4.1a) and (4.1b) we get

$$(\vec{n} \circ \langle \mathrm{grad}(p + \tfrac{1}{2}|B|^2)\rangle) = -\hat{B}_\phi^2/R_0 . \qquad (4.9)$$

The solutions of the differential equations (4.6) and (3.10) for Ψ and Φ respectively are, in cylindrical coordinates:

$$\Psi = I_\alpha(kr)\, e^{i(\alpha\phi + kz)} \qquad (4.10)$$

and

$$\Phi = i \frac{1}{R_o k^2} (k R_o \hat{b}_c + \alpha \hat{B}_\phi) I_\alpha'(kR_o) \circ M(r) e^{i(\alpha\phi+kz)} \qquad (4.11)$$

where

$$M(r) = \frac{I_\alpha'(kR_1)K_\alpha(kr) - K_\alpha'(kR_1)I_\alpha(kr)}{I_\alpha'(kR_o)K_\alpha'(kR_1) - I_\alpha'(kR_1)K_\alpha'(kR_o)} . \qquad (4.12)$$

I_α and K_α are the Bessel and Hankel functions of the first kind and of the α-order. I_α' and K_α' are the derivatives of I_α and K_α. The constants of integration in (4.11) are chosen in such a way that the two boundary conditions (3.5) and (3.6) are satisfied. In cylindrical coordinates they have the form:

$$\frac{\partial \Phi}{\partial r} = -ikb_c \xi_r - \frac{1}{R_o}\frac{\partial}{\partial \phi}\left\{\xi_r (2p + b_c^2 - \hat{b}_c^2)\right\} \text{ for } r = R_o \qquad (4.13)$$

and

$$\frac{\partial \Phi}{\partial r} = 0 \qquad \text{for } r = R_1 . \qquad (4.14)$$

Finally, from (4.10) and (4.11) we find for δW

$$\delta W = \frac{\pi L}{k^3 R_o} (I_a'(kR_o))^2 \left\{ (kR_o)^2 b_c^2 \frac{I_a(kR_o)}{I_a'(kR_o)} + \right.$$

$$\left. + (kR_o \hat{b}_c + a\hat{B}_\emptyset)^2 M(R_o) - kR_o \hat{B}_\emptyset^2 \right\} \qquad (4.15)$$

where L is the length of the cylinder. The first two terms are the contributions from the plasma and the vacuum respectively and they are always positive, while the last term is the contribution of the surface and it is always negative. This dispersion relation has been given for instance by Taylor [1] for given values of k and a. The sign of δW depends on three parameters which might be $b_c/\hat{B}_\emptyset$, $\hat{b}_c/\hat{B}_\emptyset$ and R_o/R_1.

(b) $k = 0$

From (3.8) it follows that if $a \neq 0$

$$\operatorname{div} \vec{\xi} = (\vec{B} \cdot \vec{Q}) = 0 . \qquad (4.16)$$

Therefore according to (3.9)

$$\delta W^{(i)} = 0 . \qquad (4.17)$$

The solution of (3.10) for Φ which satisfies the boundary conditions (4.13) and (4.14) is given by

$$\Phi = -i \frac{1}{R_o^a - R_1^{2a} R_o^{-a}} (r^a + R_1^{2a} r^{-a}) \cdot \hat{B}_\emptyset \cdot \xi_r(R_o) e^{ia\emptyset} \qquad (4.18)$$

where $\xi_r(R_o)$ is an arbitrary constant which might be normalized to unity. Using (4.18) we find for δW, if $k = 0$ and $a \neq 0$

$$\delta W = \pi L \hat{B}_\phi^2 \xi_r^2 \left\{ \frac{R_o^{2\alpha}+R_1^{2\alpha}}{R_1^{2\alpha}-R_o^{2\alpha}} \alpha - 1 \right\}. \tag{4.19}$$

We see that $\delta W > 0$ for all $\alpha \geq 1$ as long as $R_o/R_1 \neq 0$. If R_o/R_1 approaches zero $\delta W > 0$ for all $\alpha > 1$ and $\delta W = 0$ for $\alpha = 1$.

The case $\alpha = 0$ has to be treated separately. From (3.8) it follows that

$$\xi_r = ar \tag{4.20}$$

where a is an arbitrary constant. From the boundary condition (4.11) for Φ we get[1/]

$$\frac{\partial \Phi}{\partial r} = 0 \qquad \text{for} \quad r = R_o \,. \tag{4.21}$$

Therefore

$$\delta W^{(e)} = 0 \quad . \tag{4.22}$$

Using (4.18) we find for δW if $k = \alpha = 0$:

$$\delta W = \pi L \, a^2 \, R_o^2 \left\{ 2p(\gamma-1)+b_c^2+\hat{b}_c^2 \right\} \, . \tag{4.23}$$

Here δW is always greater than zero since $\gamma \geq 1$. Therefore the cylinder is completely stable against radial oscillations as one would expect.

From (4.19) and (4.23) we can conclude that the cylinder is completely stable against all perturbations

---

1/ In this case Φ is not a single-valued function. But Φ does not contribute to δW according to (4.20) and therefore this is not of importance for the cylinder.

which are independent of z with the exception of the case $\alpha = 1$ if there is no conductor present. In this case the equilibrium is neutral for a perturbation $\alpha = 1$.

5. <u>Calculation of δW for a Torus.</u>

(a) The solution of Laplace equation in toroidal coordinates.

In toroidal coordinates $\vec{\xi}$ can be Fourier analyzed in the form

$$\vec{\xi} = \vec{\xi}(\eta, \theta)\, e^{i(m\phi + n\theta)}, \tag{5.1}$$

The problem now splits again into two branches according as $m \neq 0$ or $m = 0$, since m corresponds to k in the cylindrical case. The case $m = 0$ will not be discussed in this paper. A close investigation has shown that the mathematical formulation is more complicated than for the case $m \neq 0$ and in addition Φ, the scalar potential of the vacuum field, is not always single-valued as it is when $m \neq 0$. However, from the nature of the perturbations and from the results derived above for the cylinder when $k = 0$ we expect that with an outer conductor the torus will be completely stable against perturbations with $m = 0$.

In the case $m \neq 0$, the equations (4.4) to (4.8) hold also for the toroidal geometry if we replace b_c and k by b and m respectively. Furthermore using (2.1a) and (2.1b) we get

$$(\vec{n} \cdot \langle \text{grad}\ (p + \tfrac{1}{2}\ |B|^2)\rangle) = - \frac{1}{R_o}\ [2p + \frac{r_o}{r^3}\ (b^2 - \hat{b}^2)]\ . \quad (5.2)$$

In this way (4.8), (3.11) and (3.7) become

$$\delta W^{(i)} = - \pi \frac{b^2}{m^2} \int_o^{2\pi} r\ \psi \frac{\partial \psi}{\partial \eta}\ d\theta \quad (5.3)$$

$$\delta W^{(e)} = \pi \int_o^{2\pi} r\ \Phi \frac{\partial \Phi}{\partial \eta}\ d\theta \quad (5.4)$$

and

$$\delta W^{(s)} = - \frac{\pi}{a} \int_o^{2\pi} (\xi_\eta)^2 [2pr^2 + \frac{r_o}{r}\ (b^2 - \hat{b}^2)]\ d\theta \quad (5.5)$$

where ξ is given according to (4.7), by

$$\xi_\eta = \frac{a\ \sinh^2 \eta}{m^2 (\cosh \eta - \cos \theta)} \frac{\partial \psi}{\partial \eta}\ . \quad (5.6)$$

The two boundary conditions (3.5) and (3.6) become

$$\frac{\partial \Phi}{\partial \eta} = \frac{im\hat{b}}{r\ \sinh \eta}\ \xi_\eta + \frac{1}{r} \frac{\partial}{\partial \theta} \left\{ r\ \xi_\eta [2p + \frac{1}{r^2}\ (b^2 - \hat{b}^2)]^{1/2} \right\} \quad (5.7)$$

for $\eta = \eta_o$ and

$$\frac{\partial \Phi}{\partial \eta} = 0 \quad (5.8)$$

for $\eta = \eta_1$.

The functions Ψ, Φ appearing in (5.3), (5.4), and in (5.6), (5.7) are, as we have seen, solutions of the Laplace equation and therefore may be made up of sums of terms of the type

$$\sqrt{\cosh\eta - \cos\theta}\left\{A_n Q^m_{n-\frac{1}{2}}(\cosh\eta) + B_n P^m_{n-\frac{1}{2}}(\cosh\eta)\right\}e^{i(m\phi+n\theta)}$$

where $P^m_{n-\frac{1}{2}}$ and $Q^m_{n-\frac{1}{2}}$ are the associated Legendre functions. As Ψ is regular at $\eta = \infty$, the axis of the discharge, it will consist of Q-functions alone; we shall choose

$$\Psi_n = \sqrt{\cosh\eta - \cos\theta}\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\, e^{in\theta} \qquad (5.9)$$

where here and in what follows the argument of the Legendre functions as well as the ϕ-variable have been suppressed for simplicity in writing. Then the corresponding exterior function will be

$$\Phi_n = \sqrt{\cosh\eta - \cos\theta}\sum_{s=-\infty}^{\infty}({}^n a_s Q^m_{s-\frac{1}{2}} + {}^n b_s P^m_{s-\frac{1}{2}})e^{is\theta}. \qquad (5.10)$$

The constants ${}^n a_s$, ${}^n b_s$ are to be determined by the boundary conditions (5.7) and (5.8). Clearly the functions (5.9) form a complete set of allowable functions, but, unfortunately, these functions are not orthogonal with respect to the mode number n (clearly they are orthogonal with respect to m and various

m-modes may be separated). This means that one must investigate not only all n-modes, but also all linear combinations. Thus one must form the integrals

$$\delta W_{n\ell} = \delta W_{n\ell}^{(i)} + \delta W_{n\ell}^{(e)} + \delta W_{n\ell}^{(s)} \tag{5.11}$$

where

$$\delta W_{n\ell}^{(i)} = -\frac{\pi b^2}{2m^2} \int_0^{2\pi} r \left\{ \Psi_n \frac{\partial \Psi^*}{\partial \eta} + \Psi_n^* \frac{\partial \Psi_\ell}{\partial \eta_z} \right\} d\theta$$

$$\delta W_{n\ell}^{(e)} = \frac{\pi}{2} \int_0^{2\pi} r \left\{ \Phi_n \, \partial_\eta \, \Phi_\ell^* + \Phi_n^* \, \partial_\eta \Phi_\ell \right\} d\theta \tag{5.12}$$

$$\delta W_{n\ell}^{(s)} = -\frac{\pi}{2} \int_0^{2\pi} \left[\frac{2pr^2}{a} + \frac{r_o(b^2-\hat{b}^2)}{ar} \right] \left[(\xi_\eta)_n (\xi_\eta^*)_\ell \right.$$

$$\left. + (\xi_\eta^*)_n (\xi_\eta)_\ell \right] d\theta$$

the star denoting complex conjugate. The necessary and sufficient condition for stability is that the matrix $(\delta W_{n\ell})$ be positive definite (or zero) for all m.

(b) The Plasma-Vacuum Boundary.

In this section we work out explicitly the boundary condition of equation (5.7). Substituting (5.6) into (5.7) gives

$$\frac{\partial \Phi}{\partial \eta} = \frac{i\hat{b}}{m} \frac{\partial \Psi}{\partial \eta} + \frac{a \sinh^2 \eta}{m^2} (\cosh \eta - \cos \theta) \frac{\partial}{\partial \theta} \left\{ \frac{\hat{B}_\theta}{(\cosh \eta - \cos\theta)^2} \frac{\partial \Psi}{\partial \eta} \right\} \tag{5.13}$$

where $\hat{B}_\theta$ is the vacuum value and, according to the

pressure balance equation (2.8), is given by

$$\hat{B}_\theta = \frac{\sqrt{b^2-\hat{b}^2}}{2a \sinh \eta} \left\{(2 \cosh \eta - 2 \cos \theta)^2 + M^2 \sinh^2 \eta\right\}^{1/2}$$

$$M^2 \overset{\text{def}}{=} \frac{8a^2 p}{b^2-\hat{b}^2} . \tag{5.14}$$

It is assumed that $b^2-\hat{b}^2 \geq 0$, which is the usual case. It can be shown that all our results are correct also when $\hat{b}^2 > b^2$, provided only that one interchange b and $\hat{b}$ so that the square roots in (5.14) and (5.21) remain real and M^2 positive. When (5.9) and (5.10) are employed we obtain the results

$$\frac{\partial \Psi_n}{\partial \eta} = \frac{\sinh \eta}{2\sqrt{\cosh \eta - \cos \theta}} \left\{ -q_n' e^{i(n-1)\theta} + (q_n + 2q_n' \cosh \eta) e^{in\theta} - q_n' e^{i(n+1)\theta} \right\} \tag{5.15}$$

and

$$\frac{\partial \Phi}{\partial \eta} = \frac{\sinh \eta}{2\sqrt{\cosh \eta - \cos \theta}} \sum_{s=-\infty}^{\infty} {}^nA_s \, e^{is\theta} \tag{5.16a}$$

$${}^nA_s \overset{\text{def}}{=} -({}^na_{s-1} q_{s-1}' + {}^nb_s p_{s-1}') + ({}^na_s q_s + {}^nb_s p_s) + 2 \cosh \, ({}^na_s q_s' + {}^nb_s p_s') - ({}^na_{s+1} q_{s+1}' + {}^nb_{s+1} p_{s+1}') . \tag{5.16b}$$

Here we have set, for short $q_n \overset{\text{def}}{=} Q^m_{n-\frac{1}{2}}$, $p_n \overset{\text{def}}{=} P^m_{n-\frac{1}{2}}$; the prime denotes differentiation with respect to the

argument, $\cosh \eta$. The index "n" in the upper left-hand corner indicates that we are fitting to the n^{th} plasma mode as given by (5.9).

The procedure now is to set (5.15) into (5.13) and to expand the right-hand side as a Fourier series. This leads to the Fourier coefficients

$$
\begin{aligned}
{}^{n}C_s = \frac{i\hat{b}}{m}\,{}^{n}G_s + \frac{a\sinh^2\eta}{2\pi m^2}\int_0^{2\pi}(\cosh\eta - \cos\theta)^{3/2}\times \\
\times\frac{\partial}{\partial\theta}\left\{\frac{\hat{B}_\theta}{(\cosh\eta-\cos\theta)^{5/2}}\sum_\ell {}^{n}G_\ell\, e^{i\theta}\right\}e^{-is\theta}\,d\theta
\end{aligned}
\qquad (5.17)
$$

where ${}^{n}G_s$ are the coefficients of the 3-term Fourier series in the brackets of (5.15). A partial integration may be performed on the right hand side of this expression and we obtain

$$
{}_{n}C_s = i\frac{\hat{b}}{m}\,{}^{n}G_s - \frac{\sinh\eta}{2\pi m^2}\sqrt{b^2-\hat{b}^2}\sum_\ell {}^{n}G_\ell\, I_{\ell s} \qquad (5.18)
$$

where $I_{\ell s}$ are the integrals

$$
\begin{aligned}
I_{\ell s} \overset{\text{def}}{=} \frac{1}{2}\int_0^{2\pi}\frac{\frac{3}{2}\sin\theta - is(\cosh\eta-\cos\theta)}{(\cosh\eta-\cos\theta)^2}\Big\{(2\cosh\eta - 2\cos\theta)^2 \\
+ M^2\sinh^2\eta\Big\}^{1/2} e^{i(\ell-s)\theta}\,d\theta\ .
\end{aligned}
\qquad (5.19)
$$

The integrals $I_{\ell s}$ can be evaluated by standard methods and we obtain

$$
\begin{aligned}
I_{\ell s} &= \frac{1}{2}\pi i M e^{-|\ell-s|\eta}\cdot(3\ell-5s) \quad \text{for } \ell \neq s \\
I_{\ell\ell} &= 2\pi i\ell\left(1-\frac{M}{2}\right).
\end{aligned}
\qquad (5.20)
$$

Setting this result into (5.18) gives the following:

$$ {}^nC_n = \frac{i}{m}\left\{ [b - \frac{n}{m}\sqrt{b^2-\hat{b}^2}\ \sinh\eta\cdot(1-\tfrac{1}{2}M)]\ [q_n+2q_n'\cosh\eta] - \frac{n}{m}\sqrt{b^2-\hat{b}^2}\ \sinh\eta\,(Me^{-\eta})\ q_n' \right\} $$

$$ {}^nC_{n\pm1} = \frac{1}{m}\left\{ [-\hat{b} + \frac{\sinh\eta}{m}\sqrt{b^2-\hat{b}^2}\left\{(n\pm1)(1-\frac{M}{2}) - \frac{n\pm4}{2}Me^{-2\eta}\right\}]q_n' + \frac{\sinh\eta}{m}\sqrt{b^2-\hat{b}^2}\ (2n\pm5)\ Me^{-\eta}(q_n+2q_n'\cosh\eta) \right\} \tag{5.21} $$

$$ {}^nC_{n\pm s} = \frac{i\ \sinh\eta}{4m^2}\sqrt{b^2-\hat{b}^2}\ (Me^{-s\eta})[(2n\pm5s)q_n\pm6q_n'\sinh\eta] $$

for $s \geq 2$.

Our boundary conditions which determine ${}^na_s$ and ${}^nb_s$ are now:

$$ {}^nA_s = {}^nC_s \qquad \text{for } \eta = \eta_o \tag{5.22a} $$

and

$$ {}^nA_s = 0 \qquad \text{for } \eta = \eta_1 \tag{5.22b} $$

where ${}^nA_s$ and ${}^nC_s$ are given by (5.16) and (5.21) respectively. For each n we have a different system of equations. In practice the infinite set of equations (5.22) will be replaced by a finite set, with s running from $-N$ to $N-1$, say; there will therefore be $4N$ equations for each n.

(c) Calculation of the δW's.

The integrals involved in the explicit calculation of the δW's given by equations (5.12) can be evaluated in

closed form. In fact, all of these integrals are of the form

$$I_\mu^\lambda = \int_0^{2\pi} \frac{e^{i\lambda\theta}\, d\theta}{(\cosh\eta - \cos\theta)^\mu} \tag{5.23}$$

where λ,μ are integers. Furthermore, λ may be taken positive without loss of generality, as the substitution $\theta \longrightarrow -\theta$ leaves the integral invariant and $I_\mu^{-\lambda} = I_\mu^\lambda$. The integrals needed are for $\mu = 1$ to 5 inclusive and are written out explicity below:

$$I_1^\lambda = \frac{2\pi}{\sinh\eta}\, e^{-\lambda\eta} \qquad I_2^\lambda = \frac{2\pi e^{-\lambda\eta}}{\sinh^2\eta}\left[\lambda+1+\frac{e^{-\eta}}{\sinh\eta}\right]$$

$$I_3^\lambda = \frac{2\pi e^{-\lambda\eta}}{2!\sinh^3\eta}\left[(\lambda+2)(\lambda+1) + \frac{3(\lambda+2)e^{-\eta}}{\sinh\eta} + \frac{3e^{-2\eta}}{\sinh^2\eta}\right]$$

$$\begin{aligned} I_4^\lambda = \frac{2\pi e^{-\lambda\eta}}{3!\sinh^4\eta}\Big[&(\lambda+3)(\lambda+2)(\lambda+1) + \frac{6(\lambda+3)(\lambda+2)e^{-\eta}}{\sinh\eta} + \\ &+ \frac{15(\lambda+3)e^{-2\eta}}{\sinh^2\eta} + \frac{15e^{-3\eta}}{\sinh^3\eta}\Big] \end{aligned} \tag{5.24}$$

$$\begin{aligned} I_5^\lambda = \frac{2\pi e^{-\lambda\eta}}{4!\sinh^5\eta}\Big[&(\lambda+4)(\lambda+3)(\lambda+2)(\lambda+1) + \\ &+ 10(\lambda+4)(\lambda+3)(\lambda+2)\frac{e^{-\eta}}{\sinh\eta} + 45(\lambda+4)(\lambda+3)\frac{e^{-2\eta}}{\sinh^2\eta} + \\ &+ 105(\lambda+4)\frac{e^{-3\eta}}{\sinh^3\eta} + 105\frac{e^{-4\eta}}{\sinh^4\eta}\Big]\,. \end{aligned}$$

Setting (5.9) into the first of (5.12) gives

$$\delta W_{n\ell}^{(i)} = -\frac{\pi b^2}{m^2}\left[q_n q_\ell I_1^{(n-\ell)} + 2\pi\delta_{\ell n}(q_n q_\ell' + q_n' q_\ell)\right] \cdot a \sinh^2\eta_o$$

or

$$\delta W_{n\ell}^{(i)} = -\frac{\pi^2 b^2}{m^2} a \sinh^2 \eta_o \left\{ 2q_\ell q_n \delta_{\ell n} + q_\ell q_n \frac{e^{-|n-\ell|\eta_o}}{\sinh \eta_o} \right\}. \quad (5.25)$$

Here $\delta_{n\ell}$ is the Kronecker symbol. Similarly, setting (5.10) and (5.16) into the second of equations (5.12) gives the result

$$\delta W_{n\ell}^{(e)} = \pi^2 a \sinh \eta_o \sum_r \sum_s ({}^n a_s q_s + {}^n b_s p_s) C_r e^{-|r-s|\eta_o}. \quad (5.26)$$

Finally, for the third of equations (5.12), we get

$$\delta W_{n\ell}^{(s)} = -\frac{\pi a^3 \sinh^6 \eta_o}{4m^4} \left\{ \left(\frac{b^2 - \hat{b}^2}{a^2} - \frac{\cosh \eta_o}{\sinh^2 \eta_o} \right) \left(8[\delta_{n\ell} q_n' q_\ell' + \right. \right.$$

$$\left. + 2[q_n q_\ell' + q_n' q_\ell] I_1^{|n-\ell|} + q_n q_\ell I_2^{|n-\ell|} \right) + 2p \sinh^2 \eta_o \left(4 q_n' q_\ell' I_3^{|n-\ell|} + \right.$$

$$\left. \left. + 2[q_n q_\ell' + q_n' q_\ell] I_4^{|n-\ell|} + q_n q_\ell I_5^{|n-\ell|} \right) \right\}. \quad (5.27)$$

It has been postulated that bending a very long thin cylinder into a torus will not alter the stability properties. That this is not so is evident from an examination of the above equations in the limit $\eta_o \longrightarrow 0$. In particular, it is quite easy to show that for $m = 1$ the $(0,0)$ and $(0,\pm 1)$ terms in the matrix all behave in the same way in the limit and thus the matrix never becomes diagonal as would be the case if a long thin torus had the same properties as a cylinder. (This result has been obtained for $\delta W^{(i)}$ and $\delta W^{(s)}$. For $\delta W^{(e)}$ the

dependence of the "constants" ${}^{n}a_{s}$ and ${}^{n}b_{s}$ on η_o is very complicated and no expansion in terms η_o seems possible. However, the numerical computations show convincingly that the contribution to the matrix elements from $\delta W^{(e)}$ behaves in the same manner as do the other contributions.) Physically this means that there is always at least one mode, namely $m = 1$, which feels the curvature. The $m = 1$ mode corresponds to a perturbation having just one wave length around the torus and this wave length is always of the same order of magnitude as the curvature.

6. Method of Computation and Results.

Codes have been written for the IBM-704 to determine the stability of a plasma for both the cylinder and the torus. For the cylinder, δW is computed directly from eq. (4.15). The Bessel functions that are needed are evaluated using a code written by Goldstein and Kresge [5] and results agree with those published by Tayler [1]. The main use of the code is to facilitate comparisons between the cylinder and torus.

For the torus, the situation is much more complicated. First the associated Legendre polynomials of both kinds and their derivatives are computed [6] and stored in the machine. Then the Fourier coefficients are found directly

from eq. (5.21), and the systems of equations given by (5.22) must be solved. Elimination methods do not take advantage of the great number of zeros in the matrix of the coefficients and so are very time consuming. They also give rise to very large errors, in some cases, due to loss of significance. A method suggested by S. Schechter [7] is used and is found to be very accurate.

The computation of $\delta W^{(i)}$ and $\delta W^{(s)}$ is entirely straightforward. For $\delta W^{(e)}$ it was observed that the computation of the double sum given in eq. (5.26) was the most time consuming part of the whole code. Instead of summing from $-N$ to $N-1$, we tried summing from $-N'$ to $-N'-1$ where $N' \leq N$. The omitted terms were found to make a very small contribution and the code could be speeded up in this way by a factor of as much as two. Our results show that fairly good indications of stability can be obtained with $N = 8$ and $N' = 6$. However, in regions where η_0 is small, the matrix elements fall off very slowly with increasing n and ℓ as can be seen by an examination of the equations, and the matrix segment with $N = N' = 19$, the maximum permitted by the code, may not be large enough. This is not a very serious limitation since these are not regions of physical interest. Once the matrix of δW is obtained the eigenvalues are computed [8] by rotating the matrix to tridiagonal form and then using a Sturm sequence of

polynomials to isolate the roots.

The computations show that for a given configuration there is a value of N (and N') such that further increase in size does not change the eigenvalues — the added terms are so small that the added eigenvalues are computed to be zero. If N (and N') are taken to be less than optimum, it has been observed that the resulting eigenvalues will be changed in the direction to make the system appear less stable. Thus the machine computations give results which are only a lower bound on the complete region of stability.

For a given m, it can be shown that δW depends on four parameters, the aspect ratio Δ, the ratio between the conductor radius and the plasma radius denoted by Λ, and two further parameters which can be chosen as B_i^2, the ratio of the "average" magnetic pressure of the toroidal field inside the plasma to the gas pressure, and $\hat{B}_e^2$, the ratio of the "average" magnetic pressure of the toroidal field in the vacuum to the gas pressure. B_e^2 and $\hat{B}_e^2$ are defined by

$$B_i^2 = b^2/(2a^2p) \qquad B_e^2 = \hat{b}^2/(2a^2p) \,. \tag{6.1}$$

Thus in computing δW we can normalize both a and p to unity.

Now in order to make comparison with the cylinder we

define similarly dimensionless field quantities for the cylinder

$$\bar{b}_c^2 = b_c^2 / 2p \qquad \hat{\bar{b}}_c^2 = \hat{b}_c^2 / 2p \quad . \tag{6.2}$$

It is convenient to use the same units as does Tayler. He defines

$$b_i = b_c / \hat{B}_\phi \qquad b_e = \hat{b}_c / \hat{B}_\phi \tag{6.3}$$

and substitution of (6.3) into (4.2) gives

$$1 + b_e^2 - b_i^2 = 2p / \hat{B}_\phi^2 \quad . \tag{6.4}$$

Since the right hand side is positive, we have the restriction

$$1 + b_e^2 \geq b_i^2 \quad . \tag{6.5}$$

To relate b_i and b_e to $\bar{b}_c$ and $\hat{\bar{b}}_c$ we set (6.2) into (4.2) giving an expression for $\hat{B}_\phi^2/p$ which is substituted into (6.4). Then after some reduction we get

$$\bar{b}_c = b_i \sqrt{1/(1+b_e^2-b_i^2)} \qquad \hat{\bar{b}}_c = b_i \sqrt{1/(1+b_e^2-b_i^2)} . \tag{6.6}$$

From (2.8) we have

$$1 + G^2 (B_i^2 - B_e^2) = \hat{B}_\theta^2 / 2p \tag{6.7}$$

where the geometric factor $G = (\cosh \eta_o - \cos \theta)/\sinh \eta_o$. In the limit as $\eta_o \longrightarrow \infty$, $G \longrightarrow 1$ but even for moderate

values of η_o, G is close to unity. The right hand side of (6.7) must be positive so we have approximately

$$1 + B_i^2 \geq B_s^2 \qquad (6.8)$$

For comparisons between cylinder and torus, we choose critical values between stability and instability of b_e and b_i and use (6.6) to get the corresponding values of $\bar{b}_c$ and $\hat{b}_c$. These are then identified with B_i^2 and B_e^2.

An examination of equations for δW (5.25-5.27) would lead one to expect that configurations which are stable for $m = 1$ will be stable for higher values of m. $\delta W^{(i)}$ and $\delta W^{(e)}$ are derived from positive definite expressions and so must have eigenvalues which are all positive. These terms vary as m^{-2}. The surface term (5.27) gives a negative contribution to δW but varies as m^{-4}. Eigenvalues of the δW matrix have been calculated for $m = 2$ and $m = 3$. The results show that if $m = 1$ is stable $m = 3$ is always stable. In a few cases $m = 2$ is not quite stable, but so close to stability that only a slight shift in one of the parameters would be sufficient to achieve stability. In general it can be stated that if $m = 1$ is stable all higher modes are also stable. The case $m = 0$ has been discussed above. The results quoted below are for $m = 1$ only.

Regions of stability have been mapped out by choosing B_i , B_e (satisfying 6.8) and $\Lambda \equiv R_1 / R_o$ and then determining what value of Δ will give a stable configuration. It has been found that a stable configuration can be achieved for very large ranges of the parameters. Roughly, for Λ as large as 5 a stable configuration exists for all B_i and $B_e \lesssim 10$. If B_i and B_e do not differ by more than ten or twenty percent the ranges of these parameters for which stability exists is extended up to 100 or more. For a given triplet B_i , B_e and Λ in these regions there is a critical Δ_c such that for all $\Delta > \Delta_c$ the configuration is unstable while for all $\Delta \leq \Delta_c$ the configuration is stable. It should be noted that in general Δ_c is small, i.e. 1.5 or less. To achieve stability for an aspect ratio as large as 3.0 requires that $B_i \simeq B_e$ (to make the negative contribution from $\delta W^{(s)}$ very small) and that Λ be close to unity.

The results can be presented in another way which gives some comparisons between torus and cylinder. From eq. (4.15) values of b_c , $\hat{b}_c$ and Λ are determined such that $\delta W = 0$ i.e. the cylinder is in neutral equilibrium for one value of k but stable for all other k. This value of k $(=k_n)$ determines a wave length. If a cylinder of just this length is bent into a torus

then the stability or instability of this toroidal configuration can be taken as an indication of the effect of the curvature. In making the correspondence between cylinder and torus, the magnetic fields are identified as explained above. The aspect ratio $\Delta = \cosh \eta_1$ is computed by identifying $(k_n R_1)$ with $(\sinh \eta_1)^{-1}$. The computations show that in most cases the toroidal configurations obtained in this way are stable. (Exceptions lie in the regions where B_i and B_e are large and not nearly the same.) It should be noted, however, that the aspect ratios in these cases are all small -- less than 1.25 for $\Lambda = 3.0$ and less than 1.75 for $\Lambda = 1.5$.

These results indicate that the effect of curvature is stabilizing when the torus is related to the cylinder in the manner discussed in the preceding paragraph. On the other hand, a thin torus, that is, one with a large aspect ratio, is never stable and in fact this study indicates that it is not possible to obtain a stable plasma in a toroidal gas discharge which has both a reasonable aspect ratio $(\Delta \gtrsim 3)$ and a value of Λ large enough to compress the gas well away from the conducting walls.

BIBLIOGRAPHY

[1] Rosenbluth, M. Los Alamos Report, No. 2030 (1957); Third International Conference on Ionization Phenomenon in Gases. Venice (1957).

Tayler, R. J., Proc. Phys. Soc. B. 70, 314 (1957).

Shafronov, V. D., Atomnaya Energiya 5, 709 (1957).

[2] Bernstein, I. B., Frieman, E. A., Kruskul, M. D., and Kulsrud, R. M., Proc. Roy. Soc. A. 244, 17 (1957).

[3] Jörgens, K., (Unpublished).

[4] Bierman, L, Hahn, K, Jürgens, K, and Lüst, R., Z. fur Naturforsehung, 12a, 826 (1957).

[5] Goldstein, M. and Kresge, M., NU BESI, Share Program Librarian, IBM, New York, New York.

[6] Rotenberg, A., Mathematics of Computation, (to be published).

[7] Schechter, S., Quasi-Tridiagonal Matrices and Type-Insensitive Difference Equations, NYO-2542 (1959). Quart. Appl. Math. (to be published).

[8] Givens, W., "Numerical Computation of the Characteristic Values of a Real Symmetric Matrix," ORNL 1574 (1954).

Dykaar, Racquel H. and LaBudde, C. Donald, "The NYU (Univac) Matrix Codes," NYO-6484 (1956).

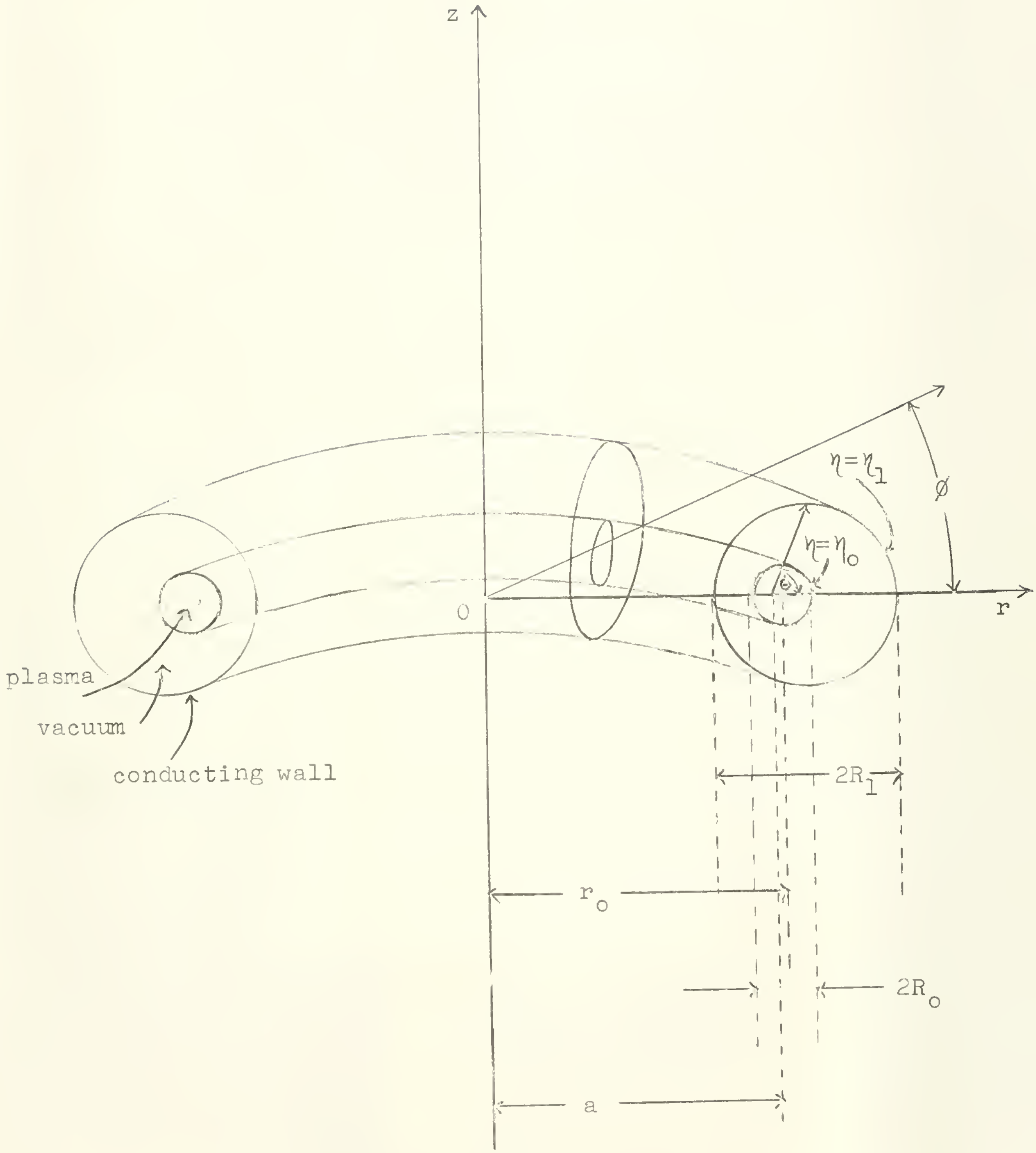
z
r
0
$\eta=\eta_1$
ϕ
$\eta=\eta_0$
θ
plasma
vacuum
conducting wall
$2R_1$
r_o
$2R_o$
a

Figure 1